Computers and Mathematics with Applications 59 (2010) 764–778

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Functions of orthogonal projectors involving the Moore–Penrose inverse

Oskar Maria Baksalary [a,*], Götz Trenkler [b]

[a] *Faculty of Physics, Adam Mickiewicz University, ul. Umultowska 85, PL 61-614 Poznań, Poland*

[b] *Department of Statistics, Dortmund University of Technology, Vogelpothsweg 87, D-44221 Dortmund, Germany*

ARTICLE INFO

Article history:
Received 15 May 2008
Received in revised form 29 March 2009
Accepted 14 April 2009

Keywords:
Orthogonal projector
Oblique projector
Partitioned matrix
Moore–Penrose inverse
Group inverse

ABSTRACT

Several results scattered in the literature express an oblique projector having given onto and along spaces in terms of a pair of orthogonal projectors. The results were established in various settings, including finite and infinite dimensional vector spaces over either real or complex numbers, but their common feature is that they are valid merely under the assumption of the nonsingularity of certain functions of the involved projectors. In the present paper, these results are unified and reestablished in a generalized form in a complex Euclidean vector space, with the generalization obtained by relaxing the nonsingularity assumption and use of the Moore–Penrose inverse instead of the ordinary inverse. Additionally, several new formulae of the type are provided.

1. Introduction

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices. The symbols $\mathbf{L}^*$, $\mathcal{R}(\mathbf{L})$, $\mathcal{N}(\mathbf{L})$, and $\mathrm{rk}(\mathbf{L})$ will stand for the conjugate transpose, column space, null space, and rank of $\mathbf{L} \in \mathbb{C}_{m,n}$, respectively. Moreover, $\mathbf{I}_n$ will be the identity matrix of order n, and for a given $\mathbf{L} \in \mathbb{C}_{n,n}$ we define $\overline{\mathbf{L}} = \mathbf{I}_n - \mathbf{L}$ (note that this notation has nothing to do with complex conjugate transpose of matrix elements).

A crucial role in the considerations of the present paper is played by orthogonal projectors in $\mathbb{C}_{n,1}$ (Hermitian idempotent matrices of order n), whose set will be denoted by $\mathbb{C}_n^{\mathrm{OP}}$, i.e.,

$$\mathbb{C}_n^{\mathrm{OP}} = \{\mathbf{L} \in \mathbb{C}_{n,n} : \mathbf{L}^2 = \mathbf{L} = \mathbf{L}^*\}.$$

An essential property of any orthogonal projector is that $\mathbf{P} \in \mathbb{C}_n^{\mathrm{OP}}$ if and only if it is expressible as $\mathbf{L}\mathbf{L}^\dagger$ for some $\mathbf{L} \in \mathbb{C}_{n,m}$, where $\mathbf{L}^\dagger \in \mathbb{C}_{m,n}$ is the Moore–Penrose inverse of $\mathbf{L}$, i.e., the unique solution to the equations

$$\mathbf{L}\mathbf{L}^\dagger\mathbf{L} = \mathbf{L}, \qquad \mathbf{L}^\dagger\mathbf{L}\mathbf{L}^\dagger = \mathbf{L}^\dagger, \qquad (\mathbf{L}\mathbf{L}^\dagger)^* = \mathbf{L}\mathbf{L}^\dagger, \qquad (\mathbf{L}^\dagger\mathbf{L})^* = \mathbf{L}^\dagger\mathbf{L}.$$

Then $\mathbf{L}\mathbf{L}^\dagger$ is the orthogonal projector onto $\mathcal{R}(\mathbf{L})$ and, consequently, $\mathbf{I}_n - \mathbf{L}\mathbf{L}^\dagger$ is the orthogonal projector onto the orthogonal complement of $\mathcal{R}(\mathbf{L})$, denoted by $\mathcal{R}(\mathbf{L})^\perp$, where $\mathbb{C}_{n,1} = \mathcal{R}(\mathbf{L}) \overset{\perp}{\oplus} \mathcal{R}(\mathbf{L})^\perp$, with the symbol $\overset{\perp}{\oplus}$ being used to indicate that the two subspaces involved in the direct sum are orthogonal. Similarly, $\mathbf{L}^\dagger\mathbf{L}$ and $\mathbf{I}_m - \mathbf{L}^\dagger\mathbf{L}$ are the orthogonal projectors onto $\mathcal{R}(\mathbf{L}^*)$ and $\mathcal{R}(\mathbf{L}^*)^\perp$, respectively, where $\mathbb{C}_{m,1} = \mathcal{R}(\mathbf{L}^*) \overset{\perp}{\oplus} \mathcal{R}(\mathbf{L}^*)^\perp$; for several facts on projectors and Moore–Penrose inverse see e.g., [1] or [2].

* Corresponding address: Department of Statistics, Dortmund University of Technology, Vogelpothsweg 87, D-44221 Dortmund, Germany. Tel.: +49 231 755 59 08; fax: +49 231 755 59 07.
E-mail addresses: baxx@amu.edu.pl (O.M. Baksalary), trenkler@statistik.uni-dortmund.de (G. Trenkler).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.10.017

Let $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ be of rank r. By *spectral theorem*, there exists a unitary $\mathbf{U} \in \mathbb{C}_{n,n}$ such that

$$\mathbf{P} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{1.1}$$

Representation (1.1) can be used to determine partitioning of any other orthogonal of order n, say $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. Namely, with the use of the same matrix $\mathbf{U}$, we can write

$$\mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^*, \tag{1.2}$$

with $\mathbf{A} \in \mathbb{C}_{r,r}$ and $\mathbf{D} \in \mathbb{C}_{n-r,n-r}$ being Hermitian. Two particular versions of representation (1.2) are obtained when $r = 0$, in which case matrices $\mathbf{A}$ and $\mathbf{B}$ are absent, and when $r = n$, in which case matrices $\mathbf{D}$ and $\mathbf{B}$ are absent. In general, matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$, involved in representation (1.2), satisfy a number of useful relationships, which are collected in Section 2.

Several results scattered in the literature express an oblique projector having given onto and along spaces in terms of a pair of orthogonal projectors. These results were established in various settings, including finite and infinite dimensional vector spaces over either real or complex numbers, but their common feature is that they are valid merely under the assumption of the nonsingularity of certain functions of the involved projectors, see e.g., [3–7]. In Section 3 these results are unified and reestablished in a generalized form in a complex Euclidean vector space, with the generalization obtained by relaxing the nonsingularity assumption and use of the Moore–Penrose inverse instead of the ordinary inverse. Additionally, several new formulae of the type are provided.

The paper is concluded with an Appendix whose aim is, besides providing several interesting formulae involving functions of $\mathbf{P}$ and $\mathbf{Q}$ considered in Section 3, to shed some additional light on the power of the formalism utilized. It should be underlined that the collection of the relationships given therein is illustrative and was chosen from a large set of possible formulae linking projectors $\mathbf{P}$ and $\mathbf{Q}$.

In what follows, the symbol $\mathbf{P}_\chi$ will denote the orthogonal projector, which projects (orthogonally) onto the subspace χ. Furthermore, with regard to the orthogonal projectors onto the column spaces of submatrices of $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ given in (1.2), we will use the convention according to which $\mathbf{P_L}$ stands for $\mathbf{P_L} = \mathbf{L}\mathbf{L}^\dagger$ and $\widetilde{\mathbf{P}}_\mathbf{L}$ for $\widetilde{\mathbf{P}}_\mathbf{L} = \mathbf{I}_k - \mathbf{L}\mathbf{L}^\dagger$, where $\mathbf{I}_k$ is the identity matrix of appropriate order and $\mathbf{L} \in \{\mathbf{A}, \overline{\mathbf{A}}, \mathbf{D}, \overline{\mathbf{D}}\}$.

2. Preliminary results

The following four lemmas concern relationships among submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ involved in matrix $\mathbf{Q}$ given in (1.2) and will be used extensively in the computations below.

Lemma 1. *Let $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ be partitioned as in (1.2). Then:*

(i) $\mathbf{A} = \mathbf{A}^2 + \mathbf{B}\mathbf{B}^*$ *or, equivalently,* $\mathbf{A}\overline{\mathbf{A}} = \mathbf{B}\mathbf{B}^*$,
(ii) $\mathbf{B} = \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{D}$ *or, equivalently,* $\mathbf{B}^* = \mathbf{B}^*\mathbf{A} + \mathbf{D}\mathbf{B}^*$,
(iii) $\mathbf{D} = \mathbf{D}^2 + \mathbf{B}^*\mathbf{B}$ *or, equivalently,* $\mathbf{D}\overline{\mathbf{D}} = \mathbf{B}^*\mathbf{B}$.

Proof. The three relationships are straightforward consequences of the condition $\mathbf{Q}^2 = \mathbf{Q}$. □

It is noteworthy that conditions (i) and (iii) of Lemma 1 combined with the facts that $\mathbf{A}$ and $\mathbf{D}$ are Hermitian, respectively, ensure that $\mathbf{A}$ and $\mathbf{D}$ are both nonnegative definite.

Lemma 2. *Let $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ be partitioned as in (1.2). Then:*

(i) $\overline{\mathbf{A}} = \overline{\mathbf{A}}^2 + \mathbf{B}\mathbf{B}^*$,	(ii) $\mathbf{B}\mathbf{D} = \overline{\mathbf{A}}\mathbf{B}$,
(iii) $\mathbf{A}\mathbf{B} = \mathbf{B}\overline{\mathbf{D}}$,	(iv) $\overline{\mathbf{D}} = \overline{\mathbf{D}}^2 + \mathbf{B}^*\mathbf{B}$,
(v) $\mathbf{A}\mathbf{A}^\dagger\mathbf{B} = \mathbf{B}$,	(vi) $\overline{\mathbf{A}}\,\overline{\mathbf{A}}^\dagger\mathbf{B} = \mathbf{B}$,
(vii) $\mathbf{D}\mathbf{D}^\dagger\mathbf{B}^* = \mathbf{B}^*$,	(viii) $\overline{\mathbf{D}}\,\overline{\mathbf{D}}^\dagger\mathbf{B}^* = \mathbf{B}^*$,
(ix) $\mathbf{A}^\dagger\mathbf{B} = \mathbf{B}\overline{\mathbf{D}}^\dagger$,	(x) $\overline{\mathbf{A}}^\dagger\mathbf{B} = \mathbf{B}\mathbf{D}^\dagger$.

Proof. Conditions (i)–(iv) follow directly from Lemma 1, while condition (v) is established on account of condition (i) of Lemma 1 by noting that

$$\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}\mathbf{A}^* + \mathbf{B}\mathbf{B}^*) = \mathcal{R}(\mathbf{A}\mathbf{A}^*) + \mathcal{R}(\mathbf{B}\mathbf{B}^*) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B}),$$

where the second equality is a consequence of the fact that $\mathbf{A}\mathbf{A}^*$ and $\mathbf{B}\mathbf{B}^*$ are both nonnegative definite. Thus, $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{A})$, which can equivalently be expressed as $\mathbf{A}\mathbf{A}^\dagger\mathbf{B} = \mathbf{B}$. The next three conditions are obtained similarly.

Further, from condition (ii) of Lemma 1, it follows that $\mathbf{A}^\dagger\mathbf{B} = \mathbf{A}^\dagger(\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{D})$. Hence, on account of $\mathbf{A}^\dagger\mathbf{A}\mathbf{B} = \mathbf{B}$, obtained from condition (v) of the lemma by taking into account that $\mathbf{A}^* = \mathbf{A}$, we get $\mathbf{A}^\dagger\mathbf{B} = \mathbf{B} + \mathbf{A}^\dagger\mathbf{B}\mathbf{D}$. In consequence, $\mathbf{B} = \mathbf{A}^\dagger\mathbf{B}\overline{\mathbf{D}}$. Postmultiplying this equation by $\overline{\mathbf{D}}^\dagger$ and utilizing condition (viii) of the lemma, which can equivalently be expressed as $\mathbf{B}\overline{\mathbf{D}}\,\overline{\mathbf{D}}^\dagger = \mathbf{B}$, we arrive at condition (ix). The last condition is established analogously. □

Lemma 3. *Let* $\mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *be partitioned as in (1.2). Then:*

$$\begin{array}{ll} \text{(i)}\ \mathbf{A} - \mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \widetilde{\mathbf{P}}_{\overline{\mathbf{A}}}, & \text{(ii)}\ \overline{\mathbf{A}} + \mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \mathbf{P}_{\overline{\mathbf{A}}}, \\ \text{(iii)}\ \mathbf{D} - \mathbf{B}^*\mathbf{A}^\dagger\mathbf{B} = \widetilde{\mathbf{P}}_{\overline{\mathbf{D}}}, & \text{(iv)}\ \overline{\mathbf{D}} + \mathbf{B}^*\mathbf{A}^\dagger\mathbf{B} = \mathbf{P}_{\overline{\mathbf{D}}}, \\ \text{(v)}\ \mathbf{D} + \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B} = \mathbf{P}_{\mathbf{D}}, & \text{(vi)}\ \overline{\mathbf{D}} - \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B} = \widetilde{\mathbf{P}}_{\mathbf{D}}, \\ \text{(vii)}\ \mathbf{A} + \mathbf{B}\overline{\mathbf{D}}^\dagger\mathbf{B}^* = \mathbf{P}_{\mathbf{A}}, & \text{(viii)}\ \overline{\mathbf{A}} - \mathbf{B}\overline{\mathbf{D}}^\dagger\mathbf{B}^* = \widetilde{\mathbf{P}}_{\mathbf{A}}. \end{array}$$

Proof. On account of conditions (i) and (x) of Lemmas 1 and 2, respectively, it follows that $\mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \overline{\mathbf{A}}^\dagger\mathbf{A}\overline{\mathbf{A}}$. Hence, $\mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \overline{\mathbf{A}}^\dagger(\mathbf{I}_r - \overline{\mathbf{A}})\overline{\mathbf{A}}$ and taking into account that $\overline{\mathbf{A}}\,\overline{\mathbf{A}}^\dagger = \overline{\mathbf{A}}^\dagger\overline{\mathbf{A}}$ (being a consequence of $\overline{\mathbf{A}} = \overline{\mathbf{A}}^*$), we in turn get $\mathbf{A} - \mathbf{B}\mathbf{D}^\dagger\mathbf{B}^* = \mathbf{I}_r - \overline{\mathbf{A}}^\dagger\overline{\mathbf{A}}$, establishing condition (i) of the lemma. The remaining seven conditions are obtained in a similar fashion. $\square$

Lemma 4. *Let* $\mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *be partitioned as in (1.2). Then:*

(i) $\mathrm{rk}(\overline{\mathbf{A}}) = r - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$,
(ii) $\mathrm{rk}(\overline{\mathbf{D}}) = n - r + \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{D})$.

Proof. From (2.12) in [8] it follows that $\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\overline{\mathbf{A}}) - r$. Hence, on account of point (i) of Lemma 1, we get

$$\mathrm{rk}(\overline{\mathbf{A}}) = r + \mathrm{rk}(\mathbf{B}\mathbf{B}^*) - \mathrm{rk}(\mathbf{A}) = r + \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{A}).$$

Condition (ii) is established in a similar way. $\square$

An important tool in constructing orthogonal projectors onto given column spaces is provided by the next lemma recalling two facts known in the literature.

Lemma 5. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$. *Then:*

(i) $\mathbf{P} + \overline{\mathbf{P}}(\overline{\mathbf{P}}\mathbf{Q})^\dagger$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})$,
(ii) $\mathbf{P} - \mathbf{P}(\mathbf{P}\overline{\mathbf{Q}})^\dagger$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$.

Proof. Conditions (i) and (ii) constitute equivalences (3.1) ⇔ (3.6) and (4.1) ⇔ (4.8) in [9], respectively. $\square$

Using Lemma 5, we obtain the following representations of the orthogonal projectors onto sums and intersections of certain subspaces, including their dimensions.

Lemma 6. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in (1.2). Then:*

(i) $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] = r + \mathrm{rk}(\mathbf{D})$,
(ii) $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})] = n + \mathrm{rk}(\mathbf{B}) - \mathrm{rk}(\mathbf{D})$,
(iii) $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] = n - r + \mathrm{rk}(\mathbf{A})$,
(iv) $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})] = n - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$.

Proof. We establish point (i) only, for the remaining ones are obtained analogously. On account of conditions (iii) and (vii) of Lemmas 1 and 2, respectively, direct verifications show that the Moore–Penrose inverse of

$$\overline{\mathbf{P}}\mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^* \tag{2.1}$$

is given by

$$(\overline{\mathbf{P}}\mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{B}\mathbf{D}^\dagger \\ \mathbf{0} & \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*. \tag{2.2}$$

Hence, from statement (i) of Lemma 5, it follows that the orthogonal projector onto $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})$ has the form claimed in point (i). The validity of the remaining part of this point is clearly seen. $\square$

Lemma 7. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in (1.2). Then:*

(i) $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \widetilde{\mathbf{P}}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$,
(ii) $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \widetilde{\mathbf{P}}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})] = r - \mathrm{rk}(\mathbf{A})$,

(iii) $\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widetilde{\mathbf{P}}_{\overline{\mathbf{D}}}\end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\mathbf{D}) - \mathrm{rk}(\mathbf{B})$,

(iv) $\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*$, *where* $\dim[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})] = n - r - \mathrm{rk}(\mathbf{D})$.

Proof. We again establish point (i) only. Direct verifications, with the use of conditions (iii) of Lemma 1, (vi), (x) of Lemma 2, and (ii) of Lemma 3, show that the Moore–Penrose inverse of

$$\mathbf{P}\overline{\mathbf{Q}} = \mathbf{U}\begin{pmatrix}\overline{\mathbf{A}} & -\mathbf{B}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^* \tag{2.3}$$

is given by

$$(\mathbf{P}\overline{\mathbf{Q}})^\dagger = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0}\\ -\mathbf{D}^\dagger\mathbf{B}^* & \mathbf{0}\end{pmatrix}\mathbf{U}^*. \tag{2.4}$$

Hence, from statement (ii) of Lemma 5, it follows that the orthogonal projector onto $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})$ is of the form given in point (i) of the lemma. Furthermore, since $\dim[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})] = \mathrm{rk}[\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}]$, it is seen that

$$\dim[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\widetilde{\mathbf{P}}_{\overline{\mathbf{A}}}) = \mathrm{rk}(\mathbf{I}_r - \overline{\mathbf{A}}\,\overline{\mathbf{A}}^\dagger) = r - \mathrm{rk}(\overline{\mathbf{A}}),$$

and the equality on the right-hand side of point (i) follows on account of condition (i) of Lemma 4. □

The theorem below provides several characterizations involving $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ expressed in terms of ranks of submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$.

Theorem 1. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$ *and let* $\mathbf{Q}$ *be partitioned as in (1.2). Then:*

(i) $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\} \;\Leftrightarrow\; \mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$,
(ii) $\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1} \;\Leftrightarrow\; \mathrm{rk}(\mathbf{D}) = n - r$,
(iii) $\mathcal{R}(\mathbf{P})\perp\mathcal{R}(\mathbf{Q}) \;\Leftrightarrow\; \mathrm{rk}(\mathbf{A}) = 0$,
(iv) $\mathcal{R}(\mathbf{P})\oplus\mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1} \;\Leftrightarrow\; \mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{B})$ *and* $\mathrm{rk}(\mathbf{D}) = n - r$,
(v) $\mathcal{R}(\mathbf{P})\overset{\perp}{\oplus}\mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1} \;\Leftrightarrow\; \mathrm{rk}(\mathbf{A}) = 0$ *and* $\mathrm{rk}(\mathbf{D}) = n - r$.

Proof. Equivalences (i) and (ii) follow directly from points (i) of Lemmas 7 and 6, respectively. To establish the next condition, we utilize the fact that $\mathcal{R}(\mathbf{P})\perp\mathcal{R}(\mathbf{Q}) \Leftrightarrow \mathbf{P}\mathbf{Q} = \mathbf{0}$. As is easy to see, $\mathbf{P}\mathbf{Q} = \mathbf{0}$ if and only if $\mathbf{A} = \mathbf{0}$, i.e., $\mathrm{rk}(\mathbf{A}) = 0$. The proof is concluded with observations that condition (iv) is obtained by combining conditions (i) and (ii), whereas condition (v) follows by combining conditions (ii) and (iii). □

3. Main results

Taking the conjugate transpose on both sides of (2.4) leads to

$$(\overline{\mathbf{Q}}\mathbf{P})^\dagger = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & -\mathbf{B}\mathbf{D}^\dagger\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*. \tag{3.1}$$

It is easily seen that condition (vi) of Lemma 2 ensures that $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ is idempotent; see [10, Lemma 2·3] and [5, p. 830]. To determine the onto and along spaces of this projector, first observe that, on account of condition (ii) of Lemma 3, the orthogonal projector onto $\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]$, i.e., $\mathbf{P}_{\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]} = (\overline{\mathbf{Q}}\mathbf{P})^\dagger\overline{\mathbf{Q}}\mathbf{P}$, is given by

$$\mathbf{P}_{\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]} = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \tag{3.2}$$

whereas, on account of conditions (iii) of Lemma 1 and (vi), (x) of Lemma 2, the orthogonal projector onto $\mathcal{N}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]$, i.e., $\mathbf{P}_{\mathcal{N}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]} = \mathbf{I}_n - \overline{\mathbf{Q}}\mathbf{P}(\overline{\mathbf{Q}}\mathbf{P})^\dagger$, has the form

$$\mathbf{P}_{\mathcal{N}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]} = \mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^* & \mathbf{D}+\widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*.$$

On the other hand, applying statement (ii) of Lemma 5 to (1.1) and the projector given in point (iv) of Lemma 6, leads to

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

while, on account of condition (vii) of Lemma 2, applying statement (i) of Lemma 5 to (1.2) and the projector given in point (iv) of Lemma 7, gives

$$\mathbf{P}_{\mathcal{R}(\mathbf{Q})\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^* & \mathbf{D}+\widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*.$$

As a consequence,

$$\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger] = \mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})], \tag{3.3}$$

$$\mathcal{N}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger] = \mathcal{R}(\mathbf{Q}) \overset{\perp}{\oplus} [\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]. \tag{3.4}$$

It follows from (3.3) and (3.4) that when $\mathcal{R}(\mathbf{P}) \oplus \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$, i.e., when $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$ and $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$, or, equivalently, $\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q}) = \{\mathbf{0}\}$ and $\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q}) = \mathbb{C}_{n,1}$, then $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ is the oblique projector onto $\mathcal{R}(\mathbf{P})$ along $\mathcal{R}(\mathbf{Q})$. This fact was also mentioned by Greville [5, p. 830], who actually restricted his considerations to the case when $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are complementary.

Another representation of oblique projectors in terms of two orthogonal projectors was established earlier by Afriat [3, Theorem 4·2]. Namely, realizing that condition $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$ ensures that $\mathbf{I}_n - \mathbf{PQ}$ is nonsingular (this fact was also stated in [5, Lemma]), he showed that $(\mathbf{I}_n - \mathbf{PQ})^{-1}\mathbf{P}\overline{\mathbf{Q}}$ is the oblique projector onto $\mathcal{R}(\mathbf{P})$ along $\mathcal{R}(\mathbf{Q}) \oplus [\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$. The next theorem generalizes this result by relaxing the disjointness assumption. Furthermore, it shows that in general, the representations derived by Greville [5, p. 830] and Afriat [3, Theorem 4·2] correspond to the same oblique projector.

Theorem 2. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$. *Then*

$$(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{PQ})^\dagger \mathbf{P}\overline{\mathbf{Q}} \tag{3.5}$$

is the oblique projector onto $\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$ *along* $\mathcal{R}(\mathbf{Q}) \overset{\perp}{\oplus} [\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$.

Proof. First observe that direct verifications with the use of conditions (vi) and (x) of Lemma 2 show that the Moore–Penrose inverse of

$$\mathbf{I}_n - \mathbf{PQ} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^* \tag{3.6}$$

has the form

$$(\mathbf{I}_n - \mathbf{PQ})^\dagger = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}}^\dagger & \mathbf{BD}^\dagger \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*. \tag{3.7}$$

In view of condition (x) of Lemma 2, postmultiplying (3.7) by $\mathbf{P}\overline{\mathbf{Q}}$ given in (2.3) leads to matrix of the form (3.1). The column and null spaces of $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ were already characterized in (3.3) and (3.4). □

An easy observation originating from Theorem 2 is that (3.5) can be rewritten as

$$(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{PQ})^\dagger \mathbf{P}(\mathbf{I}_n - \mathbf{PQ}),$$

which may be interpreted as a "generalized similarity transformation" that carries $\mathbf{P}$ into the projector $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$. It is also worth mentioning that, on account of conditions (vi) and (x) of Lemma 2, from (2.3) and (3.7) we obtain

$$\mathbf{P}\overline{\mathbf{Q}}(\mathbf{I}_n - \mathbf{PQ})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

i.e., the orthogonal projector onto $\mathcal{R}[(\overline{\mathbf{Q}}\mathbf{P})^\dagger]$ given in (3.2).

The following two corollaries are obtained as straightforward consequences of Theorem 2. The first of them corresponds to Theorem 4·2 in [3].

Corollary 1. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *be such that* $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$. *Then* $\mathbf{I}_n - \mathbf{PQ}$ *is nonsingular and* $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{PQ})^{-1}\mathbf{P}\overline{\mathbf{Q}}$ *is the oblique projector onto* $\mathcal{R}(\mathbf{P})$ *along* $\mathcal{R}(\mathbf{Q}) \overset{\perp}{\oplus} [\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$.

Corollary 2. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *be such that* $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$. *Then* $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{PQ})^\dagger\mathbf{P}\overline{\mathbf{Q}}$ *is the oblique projector onto* $\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$ *along* $\mathcal{R}(\mathbf{Q})$.

It is noteworthy that the observation by Afriat [3, Theorem 4·2] and Greville [5, Lemma] that condition $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$ is sufficient for the nonsingularity of $\mathbf{I}_n - \mathbf{PQ}$ can be extended to the equivalence. This can be seen by noticing that (3.6) and (3.7) entail

$$\mathbf{P}_{\mathcal{R}(\mathbf{I}_n - \mathbf{PQ})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*,$$

from where we have $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = n$ if and only if $\mathrm{rk}(\overline{\mathbf{A}}) = r$. Combining this equality with conditions (i) of Lemma 4 and Theorem 1 leads to the conclusion that the nonsingularity of $\mathbf{I}_n - \mathbf{PQ}$ is actually equivalent to $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$.

Another observation by Greville [5, Theorem 3] is that when $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are complementary, then not only $\mathbf{I}_n-\mathbf{PQ}$ $(=(\mathbf{I}_n-\mathbf{QP})^*)$ but also $\mathbf{P}+\mathbf{Q}-\mathbf{QP}$ is nonsingular, in which case both $(\mathbf{I}_n-\mathbf{QP})^{-1}\overline{\mathbf{Q}}$ and $\mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^{-1}$ are the oblique projectors onto $\mathcal{R}(\mathbf{P})$ along $\mathcal{R}(\mathbf{Q})$. The theorem below provides a generalization and extension of Theorem 3 in [5], with the generalization obtained by voiding the assumption that $\mathcal{R}(\mathbf{P})\oplus\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$ and with the extension contained in providing the characterizations of the corresponding column and null spaces; see also [11, Corollary 6].

Theorem 3. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$. *Then:*

(i) $(\mathbf{I}_n-\mathbf{QP})^{\dagger}\overline{\mathbf{Q}}$ *is the oblique projector onto* $\{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\}\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$ *along* $\mathcal{R}(\mathbf{Q})$,

(ii) $\mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^{\dagger}$ *is the oblique projector onto* $\mathcal{R}(\mathbf{P})$ *along* $\{\mathcal{R}(\mathbf{Q})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\}\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$.

Proof. Direct verifications with the use of conditions (ii), (iii) of Lemma 1, (i), (vi), (vii), (x) of Lemma 2, and (ii) of Lemma 3 show that the Moore–Penrose inverse of

$$(\mathbf{I}_n-\mathbf{QP})^{\dagger}\overline{\mathbf{Q}}=\mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & -\mathbf{BD}^{\dagger}\\ \mathbf{0} & \widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^* \tag{3.8}$$

is given by

$$[(\mathbf{I}_n-\mathbf{QP})^{\dagger}\overline{\mathbf{Q}}]^{\dagger}=\mathbf{U}\begin{pmatrix}\overline{\mathbf{A}} & \mathbf{0}\\ -\mathbf{B}^* & \widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*.$$

Whence,

$$\mathbf{P}_{\mathcal{R}[(\mathbf{I}_n-\mathbf{QP})^{\dagger}\overline{\mathbf{Q}}]}=\mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*,\qquad \mathbf{P}_{\mathcal{N}[(\mathbf{I}_n-\mathbf{QP})^{\dagger}\overline{\mathbf{Q}}]}=\mathbf{Q}. \tag{3.9}$$

Thus, statement (i) of the theorem will be established if we show that the matrix on the left-hand side of (3.9) represents the orthogonal projector onto $\{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\}\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$. This is indeed the case, which can be seen by applying Lemma 5 to (1.1) and the projectors given in points (iv) of Lemmas 6 and 7.

The proof of statement (ii) is obtained similarly. First observe that condition (vii) of Lemma 2 ensures that the Moore–Penrose inverse of

$$\mathbf{P}+\mathbf{Q}-\mathbf{QP}=\mathbf{U}\begin{pmatrix}\mathbf{I}_r & \mathbf{B}\\ \mathbf{0} & \mathbf{D}\end{pmatrix}\mathbf{U}^* \tag{3.10}$$

is of the form

$$(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^{\dagger}=\mathbf{U}\begin{pmatrix}\mathbf{I}_r & -\mathbf{BD}^{\dagger}\\ \mathbf{0} & \mathbf{D}^{\dagger}\end{pmatrix}\mathbf{U}^*. \tag{3.11}$$

Whence, on account of conditions (vii) of Lemma 2 and (i) of Lemma 3, we obtain

$$\mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^{\dagger}=\mathbf{U}\begin{pmatrix}\mathbf{I}_r & -\mathbf{BD}^{\dagger}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \tag{3.12}$$

and further, as can be seen by direct verifications with the use of conditions (ii), (iii) of Lemma 1, (i), (vi), (vii), (x) of Lemma 2, and (ii) of Lemma 3, we have

$$[\mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^{\dagger}]^{\dagger}=\mathbf{U}\begin{pmatrix}\overline{\mathbf{A}}+\widetilde{\mathbf{P}}_{\overline{\mathbf{A}}} & \mathbf{0}\\ -\mathbf{B}^* & \mathbf{0}\end{pmatrix}\mathbf{U}^*.$$

In consequence,

$$\mathbf{P}_{\mathcal{R}[\mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^{\dagger}]}=\mathbf{P},\qquad \mathbf{P}_{\mathcal{N}[\mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^{\dagger}]}=\mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}}-\overline{\mathbf{A}} & \mathbf{B}\\ \mathbf{B}^* & \mathbf{D}+\widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*. \tag{3.13}$$

The last step of the proof is to show that the matrix on the right-hand side of (3.13) represents the orthogonal projector onto $\{\mathcal{R}(\mathbf{Q})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\}\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$. This can be done, though, by applying Lemma 5 to (1.2) and the projectors given in points (iv) of Lemmas 6 and 7. □

Theorem 3 is supplemented by two observations. The first one is that when $\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})=\{\mathbf{0}\}$, then the projector introduced in its point (i) satisfies $(\overline{\mathbf{Q}}\mathbf{P})^{\dagger}=(\mathbf{I}_n-\mathbf{QP})^{-1}\overline{\mathbf{Q}}$. The second comment is that for the nonsingularity of $\mathbf{P}+\mathbf{Q}-\mathbf{QP}$, involved in the projector introduced in point (ii) of the theorem, it is actually necessary and sufficient that $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$. This fact, not mentioned by Greville [5], can be derived from the orthogonal projector

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*,$$

obtained in a straightforward manner from (3.10) and (3.11). Hence, $\mathrm{rk}(\mathbf{P}+\mathbf{Q}-\mathbf{QP}) = n$ if and only if $\mathrm{rk}(\mathbf{D}) = n - r$, and, thus, the claim follows on account of point (ii) of Theorem 1.

Clearly, Theorem 3 remains true if $\mathbf{P}$ and $\mathbf{Q}$ are interchanged. Taking this into account leads to the following.

Corollary 3. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then:*

$$\mathcal{R}[(\mathbf{I}_n - \mathbf{QP})^\dagger\overline{\mathbf{Q}}] = \mathcal{N}[\mathbf{Q}(\mathbf{P}+\mathbf{Q}-\mathbf{PQ})^\dagger],$$
$$\mathcal{N}[(\mathbf{I}_n - \mathbf{QP})^\dagger\overline{\mathbf{Q}}] = \mathcal{R}[\mathbf{Q}(\mathbf{P}+\mathbf{Q}-\mathbf{PQ})^\dagger].$$

In the context of Theorems 2 and 3, it is natural to inquire about the conditions ensuring that the three projectors involved therein are equal. These conditions are provided in the theorem below.

Theorem 4. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then:*

(i) $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{QP})^\dagger\overline{\mathbf{Q}} \;\Leftrightarrow\; \mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$,
(ii) $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = \mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^\dagger \;\Leftrightarrow\; \mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$,
(iii) $(\mathbf{I}_n - \mathbf{QP})^\dagger\overline{\mathbf{Q}} = \mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^\dagger \;\Leftrightarrow\; \mathcal{R}(\mathbf{P}) \oplus \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$.

Proof. From (3.1) and (3.8) it follows that $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{QP})^\dagger\overline{\mathbf{Q}}$ if and only if $\mathrm{rk}(\mathbf{D}) = n - r$. Whence, the equivalence (i) is obtained on account of point (ii) of Theorem 1. Next, (3.1) and (3.12) lead to $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = \mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{QP})^\dagger \Leftrightarrow \mathrm{rk}(\overline{\mathbf{A}}) = r$. Combining this condition with points (i) of Lemma 4 and Theorem 1 establishes equivalence (ii). Finally, from (3.8) and (3.12) it is seen that the equality on the left-hand side of equivalence (iii) is satisfied if and only if both $\overline{\mathbf{A}}$ and $\mathbf{D}$ are nonsingular. In view of point (iv) of Theorem 1, this means that $\mathcal{R}(\mathbf{P})$ and $\mathcal{R}(\mathbf{Q})$ are complementary. □

Theorem 1 in Vidav [4] provides a representation of an idempotent operator in a Hilbert space in terms of two idempotent operators, under the assumption that a certain function of these two is nonsingular; see also [12, Theorem 2.1]. In what follows, we show that this result can be generalized in our settings as well, by relaxing the nonsingularity assumption.

Theorem 5. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{PQP})^\dagger\mathbf{P}\overline{\mathbf{Q}}$.

Proof. It is clear that the Moore–Penrose inverse of

$$\mathbf{I}_n - \mathbf{PQP} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^* \tag{3.14}$$

is of the form

$$(\mathbf{I}_n - \mathbf{PQP})^\dagger = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}}^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*. \tag{3.15}$$

In view of condition (x) of Lemma 2, postmultiplying (3.15) by (2.3) gives a matrix of the form (3.1). □

The onto and along spaces of the projector $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = (\mathbf{I}_n - \mathbf{PQP})^\dagger\mathbf{P}\overline{\mathbf{Q}}$ are, in general, characterized in Theorem 2. Observe that from (3.14) it is seen that $\mathbf{I}_n - \mathbf{PQP}$ is nonsingular if and only if $\overline{\mathbf{A}}$ is nonsingular. As already shown in the proof of Theorem 4, $\mathrm{rk}(\overline{\mathbf{A}}) = r$ is equivalent to $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$, in which case the onto and along spaces of $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ are as specified in Corollary 1. (Parenthetically notice that the equivalence $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = n \Leftrightarrow \mathrm{rk}(\mathbf{I}_n - \mathbf{PQP}) = n$ follows from Lemma 3.1 in [13].) However, Vidav [4] formulated his Theorem 1 under assumptions that $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$ and $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$ (with the former condition expressed equivalently as $\|\mathbf{PQ}\| < 1$, where $\|\mathbf{.}\|$ denotes the operator norm), in which case, as already mentioned, $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ projects onto $\mathcal{R}(\mathbf{P})$ along $\mathcal{R}(\mathbf{Q})$.

An additional remark referring to Theorem 5 is that an alternative formula for $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ involving $(\mathbf{I}_n - \mathbf{PQP})^\dagger$ reads $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = \mathbf{P}(\mathbf{I}_n - \mathbf{PQP})^\dagger\overline{\mathbf{Q}}$.

The next theorem provides yet another representation of the projector $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ as a product of two functions of $\mathbf{P}$ and $\mathbf{Q}$.

Theorem 6. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then* $(\overline{\mathbf{Q}}\mathbf{P})^\dagger = \mathbf{P}(\mathbf{P} - \mathbf{Q})^\dagger$.

Proof. Direct verifications with the use of conditions (iii) of Lemma 1, (vi), (vii), (x) of Lemma 2, and (ii) of Lemma 3 show that the Moore–Penrose inverse of

$$\mathbf{P}-\mathbf{Q}=\mathbf{U}\begin{pmatrix}\overline{\mathbf{A}} & -\mathbf{B}\\ -\mathbf{B}^* & -\mathbf{D}\end{pmatrix}\mathbf{U}^* \tag{3.16}$$

is of the form

$$(\mathbf{P}-\mathbf{Q})^\dagger=\mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & -\mathbf{B}\mathbf{D}^\dagger\\ -\mathbf{D}^\dagger\mathbf{B}^* & -\mathbf{P}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*. \tag{3.17}$$

Hence, the assertion is clearly seen. □

To determine the onto and along spaces of the projector $(\overline{\mathbf{Q}}\mathbf{P})^\dagger=\mathbf{P}(\mathbf{P}-\mathbf{Q})^\dagger$ when $\mathbf{P}-\mathbf{Q}$ is nonsingular, observe that from (3.16) and (3.17) it follows that

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})}=\mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*. \tag{3.18}$$

As a consequence, $\mathbf{P}-\mathbf{Q}$ is nonsingular if and only if both $\overline{\mathbf{A}}$ and $\mathbf{D}$ are nonsingular, which was shown in the proof of Theorem 4 to be equivalent to $\mathcal{R}(\mathbf{P})\oplus\mathcal{R}(\mathbf{Q})=\mathbb{C}_{n,1}$, i.e., $(\overline{\mathbf{Q}}\mathbf{P})^\dagger=\mathbf{P}(\mathbf{P}-\mathbf{Q})^{-1}$ projects onto $\mathcal{R}(\mathbf{P})$ along $\mathcal{R}(\mathbf{Q})$. This fact follows also from Corollary 2 in [11].

In the literature one can find further formulae originally established under the assumption of the nonsingularity of a certain function of $\mathbf{P}$ and $\mathbf{Q}$, which can be generalized simply by relaxing this assumption. As examples, we consider below relationships (2.3)–(2.6) constituting Theorem 2.2 in [14], which in general concerns not necessarily orthogonal projectors. First notice that, with the use of (2.2), (3.1), and (3.17), it is easily seen that relationship (2.3) in [14], which can be expressed as

$$(\mathbf{P}-\mathbf{Q})^{-1}=(\overline{\mathbf{Q}}\mathbf{P})^\dagger-(\mathbf{Q}\overline{\mathbf{P}})^\dagger,$$

where $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ is the oblique projector onto $\mathcal{R}(\mathbf{P})$ along $\mathcal{R}(\mathbf{Q})$, and $(\mathbf{Q}\overline{\mathbf{P}})^\dagger$ is the oblique projector onto $\mathcal{N}(\mathbf{P})$ along $\mathcal{N}(\mathbf{Q})$, also remains valid when $\mathbf{P}-\mathbf{Q}$ is singular, in which case

$$(\mathbf{P}-\mathbf{Q})^\dagger=(\overline{\mathbf{Q}}\mathbf{P})^\dagger-(\mathbf{Q}\overline{\mathbf{P}})^\dagger, \tag{3.19}$$

where $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ is the oblique projector onto $\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]$ along $\mathcal{R}(\mathbf{Q})\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$, and $(\mathbf{Q}\overline{\mathbf{P}})^\dagger$ is the oblique projector onto $\mathcal{N}(\mathbf{P})\cap[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]$ along $\mathcal{N}(\mathbf{Q})\overset{\perp}{\oplus}[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]$; see also [15, Theorem 4.2].

Another characterization of $(\mathbf{P}-\mathbf{Q})^{-1}$ follows from Theorem in [16], and claims that $\mathbf{P}-\mathbf{Q}$ is invertible if and only if there exists the oblique projector $\mathbf{M}$ with $\mathcal{R}(\mathbf{M})=\mathcal{R}(\mathbf{P})$ and $\mathcal{N}(\mathbf{M})=\mathcal{R}(\mathbf{Q})$. Furthermore, if $\mathrm{rk}(\mathbf{P}-\mathbf{Q})=n$, then

$$(\mathbf{P}-\mathbf{Q})^{-1}=\mathbf{M}+\mathbf{M}^*-\mathbf{I}_n. \tag{3.20}$$

This result was later supplemented by Cheng and Tian [17, p. 538] with a remark that the projector $\mathbf{M}$ satisfying (3.20) is unique and can be expressed as $\mathbf{M}=(\overline{\mathbf{Q}}\mathbf{P})^\dagger$, leading to

$$(\mathbf{P}-\mathbf{Q})^{-1}=(\overline{\mathbf{Q}}\mathbf{P})^\dagger+(\mathbf{P}\overline{\mathbf{Q}})^\dagger-\mathbf{I}_n; \tag{3.21}$$

see also [13, Theorem 6.1] and [15, Theorem 4.3]. The present formalism shows that replacing in (3.21) $(\mathbf{P}-\mathbf{Q})^{-1}$ with $(\mathbf{P}-\mathbf{Q})^\dagger$ leads to

$$(\mathbf{P}-\mathbf{Q})^\dagger-(\overline{\mathbf{Q}}\mathbf{P})^\dagger-(\mathbf{P}\overline{\mathbf{Q}})^\dagger+\mathbf{I}_n=\mathbf{U}\begin{pmatrix}\widetilde{\mathbf{P}}_{\overline{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*. \tag{3.22}$$

An alternative form of (3.22) constitutes the next theorem.

Theorem 7. *Let* $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$(\mathbf{P}-\mathbf{Q})^\dagger=\mathbf{P}_{[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]}+(\overline{\mathbf{Q}}\mathbf{P})^\dagger+(\mathbf{P}\overline{\mathbf{Q}})^\dagger-\mathbf{I}_n.$$

Proof. The proof is limited to the observation that applying statement (i) of Lemma 5 to the orthogonal projectors given in points (i) and (iv) of Lemma 7 leads to the conclusion that the matrix on the right-hand side of (3.22) represents the orthogonal projector onto $[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]$. □

By combining Theorem 7 with (3.19), we arrive at the next corollary.

Corollary 4. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$\mathbf{P}_{[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]\overset{\perp}{\oplus}[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]} + \mathbf{P}_{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{I}_n.$$

Proof. It is sufficient to show that $(\mathbf{P}\overline{\mathbf{Q}})^\dagger + (\mathbf{Q}\overline{\mathbf{P}})^\dagger$ is the orthogonal projector onto $[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$. This fact can be seen, though, by applying statement (ii) of Lemma 5 to the orthogonal projectors given in points (i) and (iv) of Lemma 6. □

Subsequent considerations will involve the Moore–Penrose inverse of

$$\mathbf{P} + \mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r + \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^*, \tag{3.23}$$

which, as can be confirmed with the use of conditions (iii) of Lemma 1, (vi), (vii), (x) of Lemma 2, and (i), (ii) of Lemma 3, is given by

$$(\mathbf{P} + \mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - \frac{1}{2}\widetilde{\mathbf{P}}_{\overline{\mathbf{A}}} & -\mathbf{B}\mathbf{D}^\dagger \\ -\mathbf{D}^\dagger\mathbf{B}^* & 2\mathbf{D}^\dagger - \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*. \tag{3.24}$$

A direct consequence of (3.23) and (3.24) is that

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*, \tag{3.25}$$

whence it is seen that $\mathbf{P} + \mathbf{Q}$ is nonsingular if and only if $\mathrm{rk}(\mathbf{D}) = n - r$. (Parenthetically notice that the fact that the nonsingularity of $\mathbf{P} - \mathbf{Q}$ implies the nonsingularity of $\mathbf{P} + \mathbf{Q}$ was pointed out in [11, p. 391]; see also [18].)

Observe that, on account of conditions (iii) of Lemma 1 and (vi) of Lemma 2, $(\overline{\mathbf{Q}}\mathbf{P})^\dagger$ given in (3.1) satisfies

$$[\mathbf{I}_n - (\mathbf{P}\overline{\mathbf{Q}})^\dagger](\overline{\mathbf{Q}}\mathbf{P})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{D}^\dagger\mathbf{B}^* & \mathbf{P}_{\mathbf{D}} - \mathbf{D}^\dagger \end{pmatrix}\mathbf{U}^*. \tag{3.26}$$

Since $(\mathbf{P}\overline{\mathbf{Q}})^\dagger[\mathbf{I}_n - (\overline{\mathbf{Q}}\mathbf{P})^\dagger]$ is the conjugate transpose of $[\mathbf{I}_n - (\mathbf{P}\overline{\mathbf{Q}})^\dagger](\overline{\mathbf{Q}}\mathbf{P})^\dagger$, combining (3.25) and (3.26) with point (i) of Lemma 7 leads to

$$(\mathbf{P} + \mathbf{Q})^\dagger = \mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})} - \tfrac{1}{2}\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} - [\mathbf{I}_n - (\mathbf{P}\overline{\mathbf{Q}})^\dagger](\overline{\mathbf{Q}}\mathbf{P})^\dagger - (\mathbf{P}\overline{\mathbf{Q}})^\dagger[\mathbf{I}_n - (\overline{\mathbf{Q}}\mathbf{P})^\dagger]. \tag{3.27}$$

If now both $\overline{\mathbf{A}}$ and $\mathbf{D}$ are nonsingular (i.e., $\mathrm{rk}(\mathbf{P} - \mathbf{Q}) = n$, or, equivalently, $\mathcal{R}(\mathbf{P}) \oplus \mathcal{R}(\mathbf{Q}) = \mathbb{C}_{n,1}$), then (3.27) takes the form

$$(\mathbf{P} + \mathbf{Q})^{-1} = \mathbf{I}_n - [\mathbf{I}_n - (\mathbf{P}\overline{\mathbf{Q}})^\dagger](\overline{\mathbf{Q}}\mathbf{P})^\dagger - (\mathbf{P}\overline{\mathbf{Q}})^\dagger[\mathbf{I}_n - (\overline{\mathbf{Q}}\mathbf{P})^\dagger],$$

corresponding to formula (2.4) in [14]. If, on the other hand, only $\overline{\mathbf{A}}$ is nonsingular (i.e., $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$), then from (3.27) it is seen that

$$\mathbf{I}_n - [\mathbf{I}_n - (\mathbf{P}\overline{\mathbf{Q}})^\dagger](\overline{\mathbf{Q}}\mathbf{P})^\dagger - (\mathbf{P}\overline{\mathbf{Q}})^\dagger[\mathbf{I}_n - (\overline{\mathbf{Q}}\mathbf{P})^\dagger] - (\mathbf{P} + \mathbf{Q})^\dagger = \mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})}, \tag{3.28}$$

or, alternatively,

$$[\mathbf{I}_n - (\mathbf{P}\overline{\mathbf{Q}})^\dagger](\overline{\mathbf{Q}}\mathbf{P})^\dagger + (\mathbf{P}\overline{\mathbf{Q}})^\dagger[\mathbf{I}_n - (\overline{\mathbf{Q}}\mathbf{P})^\dagger] + (\mathbf{P} + \mathbf{Q})^\dagger = \mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})}, \tag{3.29}$$

with the orthogonal projectors on the right-hand sides of (3.28) and (3.29) being of the forms given in points (iv) of Lemma 7 and (i) of Lemma 6, respectively.

In what follows we continue generalizing Theorem 2.2 in [14]. The next result deals with formula (2.5) therein.

Theorem 8. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$(\mathbf{P} - \mathbf{Q})^\dagger = (\mathbf{P} + \mathbf{Q})^\dagger(\mathbf{P} - \mathbf{Q})(\mathbf{P} + \mathbf{Q})^\dagger.$$

Proof. The assertion follows from (3.16), (3.17), and (3.24), by utilizing conditions (iii) of Lemma 1, (vi), (vii), (x) of Lemma 2, and (ii) of Lemma 3. □

The next result refers to formula (2.6) in [14].

Theorem 9. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then*

$$(\mathbf{P} + \mathbf{Q})^\dagger - (\mathbf{P} - \mathbf{Q})^\dagger(\mathbf{P} + \mathbf{Q})(\mathbf{P} - \mathbf{Q})^\dagger = \tfrac{1}{2}\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}.$$

Proof. The assertion follows directly from (3.17), (3.23), (3.24), and point (i) of Lemma 7, by utilizing the same conditions as in the proof of Theorem 8, namely (iii) of Lemma 1, (vi), (vii), (x) of Lemma 2, and (ii) of Lemma 3. □

From Theorem 9, it is clearly seen that $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$ is equivalent to

$$(\mathbf{P}+\mathbf{Q})^{\dagger} = (\mathbf{P}-\mathbf{Q})^{\dagger}(\mathbf{P}+\mathbf{Q})(\mathbf{P}-\mathbf{Q})^{\dagger},$$

which generalizes formula (2.6) in [14], established under the assumption that $\mathbf{P}-\mathbf{Q}$ (so also $\mathbf{P}+\mathbf{Q}$) is nonsingular. An additional observation derived from the proofs of Theorems 8 and 9 is that

$$(\mathbf{P}+\mathbf{Q})^{\dagger}(\mathbf{P}-\mathbf{Q}) = (\mathbf{P}-\mathbf{Q})^{\dagger}(\mathbf{P}+\mathbf{Q}) = (\mathbf{P}\overline{\mathbf{Q}})^{\dagger} - (\mathbf{Q}\overline{\mathbf{P}})^{\dagger},$$

with $(\mathbf{P}\overline{\mathbf{Q}})^{\dagger}$ and $(\mathbf{Q}\overline{\mathbf{P}})^{\dagger}$ being conjugate transposes of the matrices given in (3.1) and (2.2), respectively.

Following the main stream of the interests of the present paper, we now focus our attention on the matrix

$$\mathbf{P}+\mathbf{Q}-\mathbf{I}_n = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & -\overline{\mathbf{D}} \end{pmatrix}\mathbf{U}^*. \tag{3.30}$$

Direct verifications with the use of conditions (i) of Lemma 1, (v), (viii), (ix) of Lemma 2, and (iv) of Lemma 3 confirm that the Moore–Penrose inverse of (3.30) is of the form

$$(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{A}^{\dagger}\mathbf{B} \\ \mathbf{B}^*\mathbf{A}^{\dagger} & -\mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*. \tag{3.31}$$

As a consequence, conditions (i) of Lemma 1, (iii), (v), (viii) of Lemma 2, and (iv) of Lemma 3 entail

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)} = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*. \tag{3.32}$$

Thus, $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ is nonsingular if and only if both $\mathbf{A}$ and $\overline{\mathbf{D}}$ are nonsingular. From points (ii) and (iii) of Lemma 6 it is seen that the nonsingularity of $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ is equivalent to the requirement that the orthogonal projectors onto $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})}$ and $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})}$ are both nonsingular, i.e., equal to the identity matrix. This condition corresponds to the equivalence (2) in [7].

The two theorems below generalize points (ii) and (iv) of Lemma 2.4 in [19], formulated with respect to idempotents in a Banach space, by voiding the assumption that $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ is nonsingular. Furthermore, the first of them extends point (ii) of Lemma 2.4 in [19] by involving in the conditions listed therein the Moore–Penrose inverses of $\mathbf{PQ}$ and $\mathbf{QP}$.

Theorem 10. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then:*

(i) $(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}\mathbf{P} = (\mathbf{PQ})^{\dagger}$, (ii) $\mathbf{Q}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger} = (\mathbf{PQ})^{\dagger}$,
(iii) $\mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger} = (\mathbf{QP})^{\dagger}$, (iv) $(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}\mathbf{Q} = (\mathbf{QP})^{\dagger}$.

Proof. Observe that, by utilizing conditions (i) and (v) of Lemmas 1 and 2, respectively, it can be shown that the Moore–Penrose inverse of $\mathbf{PQ}$ is of the form

$$(\mathbf{PQ})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{0} \\ \mathbf{B}^*\mathbf{A}^{\dagger} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{3.33}$$

Hence, conditions (i) and (iii) of the theorem follow directly from (1.1), (3.31), and (3.33), whereas conditions (ii) and (iv) are obtained from (iii) and (i), respectively, by interchanging $\mathbf{P}$ and $\mathbf{Q}$. □

The next theorem extends point (iv) of Lemma 2.4 in [19] by involving in the conditions listed therein the Moore–Penrose inverses of $\mathbf{PQP}$ and $\mathbf{QPQ}$.

Theorem 11. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$. *Then:*

(i) $[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^2\mathbf{P} = (\mathbf{PQP})^{\dagger} = \mathbf{P}[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^2$,
(ii) $[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^2\mathbf{Q} = (\mathbf{QPQ})^{\dagger} = \mathbf{Q}[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^2$.

Proof. On account of conditions (i) of Lemma 1 and (iv), (viii), (ix) of Lemma 2, the matrix given in (3.31) satisfies

$$[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^2\mathbf{P} = \mathbf{U}\begin{pmatrix} \mathbf{A}^{\dagger} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*.$$

Since

$$(\mathbf{PQP})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{A}^{\dagger} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

the left-hand side equality in point (i) is established. The right-hand side equality is obtained from the left-hand side one simply by taking the conjugate transpose. The proof is concluded with an observation that point (ii) of the theorem follows from its point (i) by interchanging $\mathbf{P}$ and $\mathbf{Q}$. □

From Proposition 2.3 in [19], originally stated by Kovarik [6] with respect to idempotents in a Banach space, it follows that if $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ is nonsingular, then the formula, known as the *Kovarik formula*,

$$\mathbf{K}(\mathbf{P},\mathbf{Q}) := \mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{-2}\mathbf{Q}$$

defines the unique projector onto $\mathcal{R}(\mathbf{P})$ along $\mathcal{N}(\mathbf{Q})$. Modifying Definition 2.6 in [19], of what are therein called the *generalized Kovarik elements*, by relaxing the nonsingularity assumption, let us define

$$\mathbf{K}_k(\mathbf{P},\mathbf{Q}) := \mathbf{P}[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^k\mathbf{Q}, \quad k\in\mathbb{N}. \tag{3.34}$$

In view of condition (iii) of Theorem 10, it is seen that $\mathbf{K}_1(\mathbf{P},\mathbf{Q}) = \mathbf{P}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}\mathbf{Q}$ satisfies $\mathbf{K}_1(\mathbf{P},\mathbf{Q}) = (\mathbf{QP})^{\dagger}\mathbf{Q}$, from which we obtain

$$\mathbf{K}_1(\mathbf{P},\mathbf{Q}) = (\mathbf{QP})^{\dagger}. \tag{3.35}$$

Since, conditions (i) of Lemma 1, (v) of Lemma 2, and (iii) of Lemma 3, combined with representations (1.1), (1.2), and (3.33), entail

$$\mathbf{P}_{\mathcal{R}[(\mathbf{QP})^{\dagger}]} = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\mathbf{A}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \qquad \mathbf{P}_{\mathcal{N}[(\mathbf{QP})^{\dagger}]} = \mathbf{U}\begin{pmatrix}\overline{\mathbf{A}} & -\mathbf{B}\\ -\mathbf{B}^* & \overline{\mathbf{D}}+\widetilde{\mathbf{P}}_{\overline{\mathbf{D}}}\end{pmatrix}\mathbf{U}^*, \tag{3.36}$$

it can be shown that $\mathbf{K}_1(\mathbf{P},\mathbf{Q})$ is the oblique projector onto $\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]$ along $\mathcal{N}(\mathbf{Q}) \overset{\perp}{\oplus} [\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]$.

The next theorem restates point (1) of Theorem 2.7 in [19], and shows that certain properties possessed by $\mathbf{K}_k(\mathbf{P},\mathbf{Q})$ under the assumption that $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ is nonsingular, remain valid also when this assumption is relaxed.

Theorem 12. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$ and let $\mathbf{K}_k(\mathbf{P},\mathbf{Q})$ be as defined in (3.34). Then*

$$\mathbf{K}_{2l+1}(\mathbf{P},\mathbf{Q}) = \mathbf{K}_{2(l+1)}(\mathbf{P},\mathbf{Q}), \quad l\in\mathbb{N}\cup\{0\}.$$

Proof. From (3.34) we have

$$\mathbf{K}_{2l+1}(\mathbf{P},\mathbf{Q}) = \mathbf{P}[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^{2l+1}\mathbf{Q} = \mathbf{P}[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^{2l}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}\mathbf{Q}.$$

Hence, since point (i) of Theorem 11 ensures that $\mathbf{P}$ commutes with $[(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^{2l}$, we further get

$$\mathbf{K}_{2l+1}(\mathbf{P},\mathbf{Q}) = [(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^{2l}\mathbf{K}_1(\mathbf{P},\mathbf{Q}).$$

On the other hand, again because of condition (i) of Theorem 11, we obtain

$$\mathbf{K}_{2(l+1)}(\mathbf{P},\mathbf{Q}) = [(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^{2l+2}\mathbf{PQ}. \tag{3.37}$$

Trivially, $\mathbf{PQ} = (\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)\mathbf{Q}$, and thus, utilizing the fact that Theorem 10 ensures that $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ commutes with its Moore–Penrose inverse, we can rewrite (3.37) in the form

$$\mathbf{K}_{2(l+1)}(\mathbf{P},\mathbf{Q}) = [(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^{2l}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}\mathbf{Q}. \tag{3.38}$$

In view of (3.35), condition (iv) of Theorem 10 transforms (3.38) into

$$\mathbf{K}_{2(l+1)}(\mathbf{P},\mathbf{Q}) = [(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)^{\dagger}]^{2l}\mathbf{K}_1(\mathbf{P},\mathbf{Q}),$$

completing the proof. □

Combining Theorem 12 with (3.35) leads to what follows.

Corollary 5. *Let $\mathbf{P},\mathbf{Q}\in\mathbb{C}_n^{\mathrm{OP}}$ and let $\mathbf{K}_1(\mathbf{P},\mathbf{Q})$, $\mathbf{K}_2(\mathbf{P},\mathbf{Q})$ be as defined in (3.34). Then* $\mathbf{K}_1(\mathbf{P},\mathbf{Q}) = \mathbf{K}_2(\mathbf{P},\mathbf{Q}) = (\mathbf{QP})^{\dagger}$.

The next theorem refers to the notion of the group inverse, and has no counterpart in the literature. Recall that the existence of the group inverse is restricted to square matrices only and for a given $\mathbf{L}\in\mathbb{C}_{n,n}$ it is the unique matrix $\mathbf{L}^{\#}\in\mathbb{C}_{n,n}$ satisfying the equations

$$\mathbf{L}\mathbf{L}^{\#}\mathbf{L} = \mathbf{L}, \qquad \mathbf{L}^{\#}\mathbf{L}\mathbf{L}^{\#} = \mathbf{L}^{\#}, \qquad \mathbf{L}\mathbf{L}^{\#} = \mathbf{L}^{\#}\mathbf{L}.$$

It is known that not every square matrix has a group inverse and that the necessary and sufficient condition for a given matrix $\mathbf{L}$ to have such an inverse is that it is of index one or, in other words, that $\mathrm{rk}(\mathbf{L}^2) = \mathrm{rk}(\mathbf{L})$. As can be confirmed by direct calculations with the use of condition (v) of Lemma 2, $\mathbf{PQ}$ has a group inverse and it is of the form

$$(\mathbf{PQ})^{\#} = \mathbf{U}\begin{pmatrix}\mathbf{A}^{\dagger} & (\mathbf{A}^{\dagger})^2\mathbf{B}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*. \tag{3.39}$$

An interesting conclusion derived from (3.33) and (3.39) is that $\mathbf{PQ}(\mathbf{PQ})^{\#} = (\mathbf{QP})^{\dagger}$. Another relevant characterization of $(\mathbf{PQ})^{\#}$ is given in what follows.

Theorem 13. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *and let* $\mathbf{K}_3(\mathbf{P}, \mathbf{Q}), \mathbf{K}_4(\mathbf{P}, \mathbf{Q})$ *be as defined in* (3.34). *Then* $\mathbf{K}_3(\mathbf{P}, \mathbf{Q}) = \mathbf{K}_4(\mathbf{P}, \mathbf{Q}) = (\mathbf{PQ})^{\#}$.

Proof. We will show that $\mathbf{K}_3(\mathbf{P}, \mathbf{Q}) = (\mathbf{PQ})^{\#}$ only, for by Theorem 12, $\mathbf{K}_3(\mathbf{P}, \mathbf{Q}) = \mathbf{K}_4(\mathbf{P}, \mathbf{Q})$. On account of conditions (iii) and (iv) of Theorem 10, it follows that

$$\mathbf{K}_3(\mathbf{P}, \mathbf{Q}) = (\mathbf{QP})^{\dagger}(\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger}(\mathbf{QP})^{\dagger}.$$

Hence, in view of conditions (i) of Lemma 1 and (viii) of Lemma 2, from (3.31) and (3.33) we get

$$\mathbf{K}_3(\mathbf{P}, \mathbf{Q}) = \mathbf{U}\begin{pmatrix} \mathbf{A}^{\dagger} & (\mathbf{A}^{\dagger})^2\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

i.e., the group inverse $(\mathbf{PQ})^{\#}$ given in (3.39). $\square$

The following result will be useful in further considerations.

Lemma 8. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ *and let* $\mathbf{K}_k(\mathbf{P}, \mathbf{Q})$ *be as defined in* (3.34). *Then*

$$\mathbf{K}_{2l}(\mathbf{P}, \mathbf{Q}) = \mathbf{U}\begin{pmatrix} (\mathbf{A}^{\dagger})^{l-1} & (\mathbf{A}^{\dagger})^{l}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \quad l \in \mathbb{N} \setminus \{1\}. \tag{3.40}$$

Proof. The assertion will be established by mathematical induction. First note that for $l = 2$ the validity of (3.40) is seen by combining $\mathbf{K}_4(\mathbf{P}, \mathbf{Q}) = (\mathbf{PQ})^{\#}$, being a part of Theorem 13, with (3.39). Subsequently, we will show that representation (3.40) holds for any $\mathbf{K}_{2l}(\mathbf{P}, \mathbf{Q})$ with a given $l > 2$ provided it is satisfied by $\mathbf{K}_{2(l-1)}(\mathbf{P}, \mathbf{Q})$.

Let $l > 2$. From (3.34) we get

$$\begin{aligned}\mathbf{K}_{2l}(\mathbf{P}, \mathbf{Q}) &= \mathbf{P}(\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger}[(\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger}]^{2(l-1)}(\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger}\mathbf{Q} \\ &= (\mathbf{QP})^{\dagger}[(\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger}]^{2(l-1)}(\mathbf{QP})^{\dagger},\end{aligned}$$

with the latter equality obtained on account of conditions (iii) and (iv) of Theorem 10. Since, $(\mathbf{QP})^{\dagger} = (\mathbf{QP})^{\dagger}\mathbf{Q} = \mathbf{P}(\mathbf{QP})^{\dagger}$, it further follows that

$$\mathbf{K}_{2l}(\mathbf{P}, \mathbf{Q}) = (\mathbf{QP})^{\dagger}\mathbf{Q}[(\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger}]^{2(l-1)}\mathbf{P}(\mathbf{QP})^{\dagger} = (\mathbf{QP})^{\dagger}\mathbf{K}^*_{2(l-1)}(\mathbf{P}, \mathbf{Q})(\mathbf{QP})^{\dagger}.$$

Hence, utilizing (3.33) and (3.40) leads to

$$\mathbf{K}_{2l}(\mathbf{P}, \mathbf{Q}) = \mathbf{U}\begin{pmatrix} \mathbf{P_A} & \mathbf{A}^{\dagger}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} (\mathbf{A}^{\dagger})^{l-2} & \mathbf{0} \\ \mathbf{B}^*(\mathbf{A}^{\dagger})^{l-1} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{P_A} & \mathbf{A}^{\dagger}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

from which, after some rearrangements with the use of condition (i) of Lemma 1, representation (3.40) follows. $\square$

Observe that combining Lemma 8 with Theorem 12 leads to

$$\mathbf{K}_{2l+1}(\mathbf{P}, \mathbf{Q}) = \mathbf{K}_{2(l+1)}(\mathbf{P}, \mathbf{Q}) = \mathbf{U}\begin{pmatrix} (\mathbf{A}^{\dagger})^{l} & (\mathbf{A}^{\dagger})^{l+1}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \quad l \in \mathbb{N}.$$

Furthermore, in view of (3.33), equalities

$$\mathbf{K}_3(\mathbf{P}, \mathbf{Q}) = \mathbf{K}_4(\mathbf{P}, \mathbf{Q}) = \cdots = (\mathbf{QP})^{\dagger} \tag{3.41}$$

are satisfied if and only if $\mathbf{P_A} = (\mathbf{A}^{\dagger})^l$ and $(\mathbf{A}^{\dagger})^{l+1}\mathbf{B} = \mathbf{A}^{\dagger}\mathbf{B}$ for all $l \in \mathbb{N}$. For each $l \in \mathbb{N}$, $\mathbf{P_A} = (\mathbf{A}^{\dagger})^l$ implies that $\mathbf{A}$, being Hermitian, is idempotent, i.e., $\mathbf{A} \in \mathbb{C}_r^{\mathsf{OP}}$. In view of the condition (i) of Lemma 1, this means that $\mathbf{B} = \mathbf{0}$, and, since the reverse implication is clearly satisfied, we conclude that equalities (3.41) hold if and only if $\mathbf{B} = \mathbf{0}$.

As already mentioned, $\mathbf{K}_1(\mathbf{P}, \mathbf{Q})$ $(= \mathbf{K}_2(\mathbf{P}, \mathbf{Q}))$ is the oblique projector onto $\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})]$ along $\mathcal{N}(\mathbf{Q}) \overset{\perp}{\oplus} [\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$. Interestingly, if $k > 2$, then $\mathbf{K}_k(\mathbf{P}, \mathbf{Q})$ are no longer idempotent, but they all have the same column and null spaces as $\mathbf{K}_1(\mathbf{P}, \mathbf{Q})$. To justify the first part of this claim, simply observe that from (3.40) it follows that

$$\mathbf{K}^2_{2(l+1)}(\mathbf{P}, \mathbf{Q}) = \mathbf{K}_{2(2l+1)}(\mathbf{P}, \mathbf{Q}), \quad l \in \mathbb{N}.$$

The justification of the second part requires the formula for the Moore–Penrose inverse of $\mathbf{K}_{2(l+1)}(\mathbf{P}, \mathbf{Q})$. As can be verified with the use of conditions (i) of Lemma 1, (v) of Lemma 2, and (iii) of Lemma 3,

$$\mathbf{K}^{\dagger}_{2(l+1)}(\mathbf{P}, \mathbf{Q}) = \mathbf{U}\begin{pmatrix} \mathbf{A}^{l+1} & \mathbf{0} \\ \mathbf{B}^*\mathbf{A}^{l} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \quad l \in \mathbb{N}. \tag{3.42}$$

Hence, straightforward calculations lead to the conclusion that the orthogonal projectors onto $\mathcal{R}[\mathbf{K}_{2(l+1)}(\mathbf{P}, \mathbf{Q})]$ and $\mathcal{N}[\mathbf{K}_{2(l+1)}(\mathbf{P}, \mathbf{Q})]$ are of the same forms as, respectively, the orthogonal projectors onto $\mathcal{R}[(\mathbf{QP})^{\dagger}] = \mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})]$ and $\mathcal{N}[(\mathbf{QP})^{\dagger}] = \mathcal{N}(\mathbf{Q}) \overset{\perp}{\oplus} [\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$ given in (3.36).

An additional observation referring to (3.42) is that, as can be shown by mathematical induction, $\mathbf{K}^{\dagger}_{2(l+1)}(\mathbf{P},\mathbf{Q}) = (\mathbf{QP})^{l+1}$ for every $l \in \mathbb{N}$.

In what follows some results of Giol [7], which were derived for idempotents in Banach spaces, are generalized by relaxing the assumption that $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ is nonsingular. The next theorem provides modified versions of the two conditions constituting Proposition 2.6 in [7].

Theorem 14. *Let $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ and let $\mathbf{K}_2(\mathbf{P},\mathbf{Q})$ be as defined in (3.34). Then:*

(i) $\mathbf{K}_2(\mathbf{P},\mathbf{Q}) + \mathbf{K}_2(\overline{\mathbf{Q}},\overline{\mathbf{P}}) = \mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)}$,
(ii) $\mathbf{K}_2[\mathbf{K}_2(\mathbf{Q},\mathbf{P}),\mathbf{K}_2(\mathbf{P},\mathbf{Q})] = \mathbf{P}_{\mathcal{R}(\mathbf{QP})}$.

Proof. First verify with the use of conditions (i) of Lemma 1, (iii), (v), (vi), (viii) of Lemma 2, and (iv) of Lemma 3 that the Moore–Penrose inverse of

$$\overline{\mathbf{P}}\,\overline{\mathbf{Q}} = \mathbf{U}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ -\mathbf{B}^* & \overline{\mathbf{D}}\end{pmatrix}\mathbf{U}^*$$

is given by

$$(\overline{\mathbf{P}}\,\overline{\mathbf{Q}})^{\dagger} = \mathbf{U}\begin{pmatrix}\mathbf{0} & -\mathbf{A}^{\dagger}\mathbf{B}\\ \mathbf{0} & \mathbf{P}_{\overline{\mathbf{D}}}\end{pmatrix}\mathbf{U}^*. \tag{3.43}$$

Hence, from (3.32), (3.33), and (3.43) it is seen that equality

$$(\mathbf{QP})^{\dagger} + (\overline{\mathbf{P}}\,\overline{\mathbf{Q}})^{\dagger} = \mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)}$$

is necessarily satisfied, and thus, in view of Corollary 5, condition (i) of the theorem is established.

For the proof of condition (ii) note that combining (3.34) with Corollary 5 leads to

$$\mathbf{K}_2[\mathbf{K}_2(\mathbf{Q},\mathbf{P}),\mathbf{K}_2(\mathbf{P},\mathbf{Q})] = (\mathbf{PQ})^{\dagger}\{[(\mathbf{PQ})^{\dagger} + (\mathbf{QP})^{\dagger} - \mathbf{I}_n]^{\dagger}\}^2(\mathbf{QP})^{\dagger}.$$

From (3.33) we obtain

$$(\mathbf{PQ})^{\dagger} + (\mathbf{QP})^{\dagger} - \mathbf{I}_n = \mathbf{U}\begin{pmatrix}2\mathbf{P}_{\mathbf{A}} - \mathbf{I}_r & \mathbf{A}^{\dagger}\mathbf{B}\\ \mathbf{B}^*\mathbf{A}^{\dagger} & -\mathbf{I}_{n-r}\end{pmatrix}\mathbf{U}^*,$$

whence, as can be verified with the use of conditions (i) of Lemma 1, (v), (viii), (ix) of Lemma 2, and (iii) of Lemma 3,

$$[(\mathbf{PQ})^{\dagger} + (\mathbf{QP})^{\dagger} - \mathbf{I}_n]^{\dagger} = \mathbf{U}\begin{pmatrix}\mathbf{A} - \widetilde{\mathbf{P}}_{\mathbf{A}} & \mathbf{B}\\ \mathbf{B}^* & -(\overline{\mathbf{D}} + \widetilde{\mathbf{P}}_{\overline{\mathbf{D}}})\end{pmatrix}\mathbf{U}^*.$$

Furthermore, in view of conditions (i) of Lemma 1 and (iii), (iv), (v), (viii) of Lemma 2,

$$\{[(\mathbf{PQ})^{\dagger} + (\mathbf{QP})^{\dagger} - \mathbf{I}_n]^{\dagger}\}^2 = \mathbf{U}\begin{pmatrix}\mathbf{A} + \widetilde{\mathbf{P}}_{\mathbf{A}} & \mathbf{0}\\ \mathbf{0} & \overline{\mathbf{D}} + \widetilde{\mathbf{P}}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*,$$

and, in consequence, by utilizing conditions (v) of Lemma 2 and (iv) of Lemma 3, we arrive at

$$\mathbf{K}_2[\mathbf{K}_2(\mathbf{Q},\mathbf{P}),\mathbf{K}_2(\mathbf{P},\mathbf{Q})] = \mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^* & \mathbf{P}_{\overline{\mathbf{D}}} - \overline{\mathbf{D}}\end{pmatrix}\mathbf{U}^*. \tag{3.44}$$

The proof is concluded with an observation that the matrix on the right-hand side of (3.44) is equal to $\mathbf{P}_{\mathcal{R}(\mathbf{QP})} = \mathbf{QP}(\mathbf{QP})^{\dagger}$. □

From the proof of condition (ii) of Theorem 14, we obtain the following.

Corollary 6. *Let $\mathbf{P},\mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ and let $\mathbf{K}_1(\mathbf{P},\mathbf{Q})$ be as defined in (3.34). Then $\mathbf{K}_1[\mathbf{K}_1(\mathbf{Q},\mathbf{P}),\mathbf{K}_1(\mathbf{P},\mathbf{Q})] = \mathbf{P}_{\mathcal{R}(\mathbf{QP})}$.*

Clearly, condition (i) of Theorem 14 leads to the equivalence

$$\mathbf{K}_2(\mathbf{P},\mathbf{Q}) + \mathbf{K}_2(\overline{\mathbf{Q}},\overline{\mathbf{P}}) = \mathbf{I}_n \;\Leftrightarrow\; \mathrm{rk}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n) = n, \tag{3.45}$$

where the equality on the left-hand side of (3.45) corresponds to the one given in point (1) of Proposition 2.6 in [7]. A similar remark concerns also condition (ii) of Theorem 14, which takes the form corresponding to the formula given in point (2) of Proposition 2.6, namely

$$\mathbf{K}_2[\mathbf{K}_2(\mathbf{Q},\mathbf{P}),\mathbf{K}_2(\mathbf{P},\mathbf{Q})] = \mathbf{Q},$$

if and only if $\mathbf{P}_{\overline{\mathbf{D}}} - \overline{\mathbf{D}} = \mathbf{D}$, or, in other words, $\mathrm{rk}(\overline{\mathbf{D}}) = n - r$. However, as already mentioned, the nonsingularity of $\mathbf{P}+\mathbf{Q}-\mathbf{I}_n$ implies the nonsingularity of $\overline{\mathbf{D}}$.

The last theorem of the paper provides a generalization of Proposition 3.2 in [7].

Theorem 15. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathrm{OP}}$ *and let* $\mathbf{K}_2(\mathbf{P}, \mathbf{Q})$ *be as defined in* (3.34). *Then*

$$\mathbf{K}_2(\mathbf{P}, \mathbf{Q}) + \mathbf{K}_2(\mathbf{Q}, \mathbf{P}) = (\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger} + \mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)}. \tag{3.46}$$

Proof. On account of Corollary 5, the assertion follows straightforwardly from (3.31)–(3.33). (Parenthetically notice that Corollary 5 ensures also that $\mathbf{K}_2(\mathbf{P}, \mathbf{Q})$ and/or $\mathbf{K}_2(\mathbf{Q}, \mathbf{P})$ in (3.46) can be replaced by $\mathbf{K}_1(\mathbf{P}, \mathbf{Q})$ and $\mathbf{K}_1(\mathbf{Q}, \mathbf{P})$, respectively.) □

It is clearly seen that if $\mathbf{P} + \mathbf{Q} - \mathbf{I}_n$ is nonsingular, then (3.46) takes the form

$$\mathbf{K}_2(\mathbf{P}, \mathbf{Q}) + \mathbf{K}_2(\mathbf{Q}, \mathbf{P}) = (\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{-1} + \mathbf{I}_n,$$

corresponding to the formula given in Proposition 3.2 in [7]. An additional observation is obtained by combining condition (i) of Theorem 14 and (3.46), namely that

$$\mathbf{K}_2(\mathbf{Q}, \mathbf{P}) - \mathbf{K}_2(\overline{\mathbf{Q}}, \overline{\mathbf{P}}) = (\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger},$$

or, equivalently,

$$(\mathbf{P}\mathbf{Q})^{\dagger} - (\overline{\mathbf{P}}\,\overline{\mathbf{Q}})^{\dagger} = (\mathbf{P} + \mathbf{Q} - \mathbf{I}_n)^{\dagger}.$$

Acknowledgements

The authors are grateful to an anonymous referee for his/her valuable remarks on an earlier version of the paper. Oskar Maria Baksalary would like to express his sincere thanks to the Alexander von Humboldt Foundation for its financial support.

Appendix

An additional light on the power of the formalism utilized in the present paper is shed below by providing several formulae involving functions of $\mathbf{P}$ and $\mathbf{Q}$ considered in Section 3. It should be underlined that the collection of the relationships is illustrative only and was chosen from a large set of possible formulae linking projectors $\mathbf{P}$ and $\mathbf{Q}$.

Let $\mathbf{L} = (\overline{\mathbf{Q}}\mathbf{P})^{\dagger} + (\mathbf{P}\overline{\mathbf{Q}})^{\dagger} - \mathbf{I}_n$, with $(\overline{\mathbf{Q}}\mathbf{P})^{\dagger}$ given in (3.1), i.e.,

$$\mathbf{L} = \mathbf{U}\begin{pmatrix} 2\mathbf{P}_{\overline{\mathbf{A}}} - \mathbf{I}_r & -\mathbf{B}\mathbf{D}^{\dagger} \\ -\mathbf{D}^{\dagger}\mathbf{B}^* & -\mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*.$$

Hence, with the use of conditions (iii) of Lemma 1, (vi), (vii), (x) of Lemma 2, and (ii) of Lemma 3, it can be shown that

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{P}_{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = (\mathbf{P} - \mathbf{Q})\mathbf{L}, \tag{A.1}$$

$$\mathbf{P} + \mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = (\mathbf{P} + \mathbf{Q})[2(\mathbf{P}\overline{\mathbf{Q}})^{\dagger}(\overline{\mathbf{Q}}\mathbf{P})^{\dagger} - \mathbf{L}], \tag{A.2}$$

with the orthogonal projectors onto $\mathcal{R}(\mathbf{P} - \mathbf{Q})$ and $\mathcal{R}(\mathbf{P} + \mathbf{Q})$ provided in (3.18) and (3.25), respectively, and those onto the intersections of subspaces obtained by applying statement (ii) of Lemma 5 to the projectors given in points (i) and (iv) of Lemma 6. Moreover, if $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$, then

$$\mathbf{P} - \mathbf{Q} = (\mathbf{P} + \mathbf{Q})\mathbf{L}(\mathbf{P} + \mathbf{Q}). \tag{A.3}$$

Similarly, with the use of the results given in the paper, one can verify that:

$$\mathbf{Q}(\mathbf{P}\overline{\mathbf{Q}})^{\dagger} = (\overline{\mathbf{Q}}\mathbf{P})^{\dagger}\mathbf{Q} = \mathbf{0}, \qquad \mathbf{P}(\overline{\mathbf{Q}}\mathbf{P})^{\dagger} = (\overline{\mathbf{Q}}\mathbf{P})^{\dagger}, \qquad (\mathbf{P}\overline{\mathbf{Q}})^{\dagger}\mathbf{P} = (\mathbf{P}\overline{\mathbf{Q}})^{\dagger}, \tag{A.4}$$

$$(\mathbf{P} + \mathbf{Q})(\mathbf{P} - \mathbf{Q})^{\dagger} + (\mathbf{P} - \mathbf{Q})^{\dagger}(\mathbf{P} + \mathbf{Q}) = 2(\mathbf{P} - \mathbf{Q})^{\dagger}, \tag{A.5}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{P}(\mathbf{P}\overline{\mathbf{Q}})^{\dagger} = (\overline{\mathbf{Q}}\mathbf{P})^{\dagger}\mathbf{P}, \tag{A.6}$$

$$\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]} = \overline{\mathbf{Q}}(\overline{\mathbf{Q}}\mathbf{P})^{\dagger} = (\mathbf{P}\overline{\mathbf{Q}})^{\dagger}\overline{\mathbf{Q}}, \tag{A.7}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = [(\mathbf{P} + \mathbf{Q})(\mathbf{P} - \mathbf{Q})^{\dagger}]^2, \tag{A.8}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = (\mathbf{P}\overline{\mathbf{Q}})^{\dagger} + (\mathbf{Q}\overline{\mathbf{P}})^{\dagger} = (\overline{\mathbf{Q}}\mathbf{P})^{\dagger} + (\overline{\mathbf{P}}\mathbf{Q})^{\dagger}, \tag{A.9}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{P}_{\mathcal{R}(\mathbf{P}\overline{\mathbf{Q}})} + \mathbf{P}_{\mathcal{R}(\overline{\mathbf{P}}\mathbf{Q})}, \tag{A.10}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} + \mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} - \mathbf{I}_n, \tag{A.11}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}, \tag{A.12}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})} - \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}, \tag{A.13}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{P}_{\mathcal{R}(\mathbf{I}_n-\mathbf{P}\mathbf{Q})}\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q})}, \tag{A.14}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{P}_{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}, \tag{A.15}$$

$$\mathbf{P}_{\mathcal{N}(\mathbf{I}_n-\mathbf{P}\mathbf{Q})} + \mathbf{P}_{\mathcal{N}(\mathbf{P}+\mathbf{Q})} = \mathbf{P}_{[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]+[\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]}, \tag{A.16}$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}+\mathbf{Q}-\mathbf{I}_n)} = \mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} + \mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} - \mathbf{I}_n. \tag{A.17}$$

References

[1] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, 2nd ed., Springer-Verlag, New York, 2003.
[2] T.L. Boullion, P.L. Odell, Generalized Inverse Matrices, Wiley-Interscience, New York, 1971.
[3] S.N. Afriat, Orthogonal and oblique projectors and the characteristics of pairs of vector spaces, Proc. Cambridge Philos. Soc. 53 (1957) 800–816.
[4] I. Vidav, On idempotent operators in a Hilbert space, Publ. Inst. Math. 4 (1964) 157–163.
[5] T.N.E. Greville, Solutions of the matrix equation $XAX = X$, and relations between oblique and orthogonal projectors, SIAM J. Appl. Math. 26 (1974) 828–832.
[6] Z.V. Kovarik, Similarity and interpolation between projectors, Acta Sci. Math. 39 (1977) 341–351.
[7] J. Giol, From a formula of Kovarik to the parametrization of idempotents in Banach algebras, Illinois J. Math. 51 (2007) 429–444.
[8] Y. Tian, G.P.H. Styan, Rank equalities for idempotent and involutory matrices, Linear Algebra Appl. 335 (2001) 101–117.
[9] R. Piziak, P.L. Odell, R. Hahn, Constructing projections on sums and intersections, Comput. Math. Appl. 37 (1999) 67–74.
[10] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Phil. Soc. 51 (1955) 406–413.
[11] J. Groß, G. Trenkler, Nonsingularity of the difference of two oblique projectors, SIAM J. Matrix Anal. Appl. 21 (1999) 390–395.
[12] J.J. Koliha, Range projections of idempotents in C^* algebras, Demonstratio Math. 34 (2001) 91–103.
[13] J.J. Koliha, V. Rakočević, Invertibility of the difference of idempotents, Linear Multilinear Algebra 51 (2003) 97–110.
[14] J.J. Koliha, V. Rakočević, I. Straškraba, The difference and sum of projectors, Linear Algebra Appl. 388 (2004) 279–288.
[15] J.J. Koliha, V. Rakočević, Fredholm properties of the difference of orthogonal projections in a Hilbert space, Integral Equations Operator Theory 52 (2005) 125–134.
[16] D. Buckholtz, Inverting the difference of Hilbert space projections, Amer. Math. Monthly 104 (1997) 60–61.
[17] S. Cheng, Y. Tian, Moore–Penrose inverse of products and differences of orthogonal projectors, Acta Sci. Math. (Szeged) 69 (2003) 533–542.
[18] J.K. Baksalary, O.M. Baksalary, Nonsingularity of linear combinations of idempotent matrices, Linear Algebra Appl. 388 (2004) 25–29.
[19] Y.-N. Chen, H.-K. Du, Y.-F. Pang, A simplification of the Kovarik formula, J. Math. Anal. Appl. 331 (2007) 13–20.